NON-LINEAR MODELS

Non-linear models are not a very recent phenomenon in demography. Lee (1974) introduced a formal model which is able to describe, among others, the feedback mechanisms specified by Easterlin. For some parameter values (i.e. when the feedback mechanisms are strong enough), the model produces sustained cycles in births. Contrary to linear stable population models, which produce cycles in births and age structures that vanish as the system reaches the stable situation, non-linear models such as the one proposed by Lee generate a persistent oscillating behaviour, even in the fertility rates. However, only recently the link was made between non-linear models and unpredictable behaviour of the endogenous variables.

The idea, which stems from weather forecasting, is that important parts of reality are inherently non-linear. Some non-linear systems behave erratically in certain critical areas of their parameter space. Such systems may display stable equilibrium behaviour, but once their parameters have surpassed so-called bifurcation points, the behaviour becomes chaotic, identification becomes impossible, and hence the models of such systems cannot be used for prediction purposes.

An illuminating example is provided by the simple logistic difference equation $x_{t+1}=\alpha.x_t(1-x_t)$, for $0<x_t<1$, and $1<\alpha<4$. For $1<\alpha<3$, x_t tends to a single stable equilibrium $(\alpha-1)/\alpha$; for $\alpha>3$, x_t becomes periodic, first with two stable points, then with four stable points at $\alpha=1+\sqrt{6}=3.44949...$, and next with 8, 16, ... equilibrium values for higher α-values. Around $\alpha=3.57$, chaos ensues (except for some particular α-values generating more regular behaviour): we observe an unbounded number of stable points with different periodicities and an unbounded number of different periodic cycles[4]. In addition, there are an uncountable number of initial points x_0 whose trajectories are totally aperiodic.

It is not a possible random component in the logistic model, and other non-linear models, that causes chaotic behaviour, but it is the non-linear nature of these deterministic models. Thus, in spite of the fact that the model is fully specified, and that all the initial conditions are known, its behaviour cannot be predicted. This view involves a dramatic shift in theoretical thinking, perhaps even a change of paradigm. The current paradigm states that our understanding of many of the social phenomena we observe, including demographic behaviour, can be enhanced with more research. Our limited ability to predict demographic and other social behaviour is only temporary. The situation will improve when we put more effort into social science research. Until then, randomness accounts for the ignorance of true underlying mechanisms. This paradigm dates back to Laplace who stated already that for him probability arises from ignorance of true causes. The same attitude was implicit in other writings in the eighteenth century (Simpson, Lagrange, Gauss), and it persisted in the nineteenth century. Determinism, as opposed to probability, would be possible when all causes are known and their effects can be predicted. Indeed, the real test of determinism is predictability (Suppes, 1984, 31).

Contrary to this paradigm, the revolutionary new view connected to non-linear models is that very simple deterministic systems, for which we know the entire specification, can produce random behaviour, and thus cannot be predicted. The view that the limited predictability of certain processes is inherent to the phenomena involved, and not merely a consequence of our (hopefully temporary) ignorance, has slowly gained acceptance in such diverse fields as meteorology, population biology, economics, but also in specialized studies such as that into the dripping faucet, or the growth of a snowflake.

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4. This model can be analysed numerically with a programmable calculator, or in a spreadsheet. Choose an appropriate initial value for x_0, for example 0.25, and observe that when the parameter α is 2, say, the model will produce a stable equilibrium value of $x=0.5$. When α is increased, so does the equilibrium value: with $\alpha=2.5$ we find $x=0.6$. But then suddenly, at α-values beyond 3, the system doesn't produce a stable equilibrium value any longer, but bounces indefinitely long between *two* values: 0.764567 and 0.538014 for $\alpha=3.1$. The model produces a stable cycle between these two values. By turning up α even more, *four* equilibrium points appear at the next *bifurcation point* around $\alpha=3.45$: 0.852443, 0.433954, 0.847451, and 0.446009. And next we observe 8 equilibrium values at $\alpha=3.54409$. For higher α-values we will need a computer programme, because the bifurcation points come faster, and we need more iterations to obtain equilibrium. But we will see 16, 32, ... equilibrium values – and suddenly, beyond $\alpha=3.569946...$ the system breaks down and produces chaos. The difference between subsequent bifurcation points decreases in the limit with the same constant factor, namely 4.6692016. This constant, called Feigenbaum's number, is universal for all bifurcation processes. A very readable analytical account of the behaviour of this logistic model is given by Mickens (1990, 275–280).

In demography, the study of non-linear models connected to chaotic behaviour is relatively new. Land (1986, 897) and Land and Schneider (1987, 17) discuss some of the aspects involved. Bonneuil (1990) constructs a non-linear model that replicates Coale's I_f index for the Pays de Caux during the years 1589–1700, and shows that mortality conditions exhibit a bifurcation point for the fertility index. The sustained cyclical behaviour, or "limit cycles" generated by the model of Lee (1974) and other fertility models (see, for instance, Frauenthal and Swick, 1983; Feichtinger and Sorger, 1990; Wachter and Lee, 1989) may be considered as a weaker form of chaos. Wachter (1991) proves that the existence of a pre-procreative age-span is a necessary condition for bifurcation points in age-specific models of population renewal. Bonneuil (1989) investigates the shifts in fertility levels in nine European countries between the 1930s and the 1960s. Day et al. (1989) present an extensive non-linear model in which fertility and population size depend on such household factors as income, consumption, preference, and cost of childrearing, and they derive conditions under which sustained cycles and chaotic behaviour emerge.

Few non-linear demographic models displaying sustained cycles or chaotic behaviour are known outside the field of fertility. A notable exception is Courgeau (forthcoming a, 1991), who studies an interregional migration model which is based upon the assumption that the migration between regions i and j in some time interval (t_0, t_1) can be described by means of a migration parameter defined as $M_{ij}/(P_i(t_0).\ P_j(t_1))$. Here M_{ij} is the number of migrants between regions i and j during the particular interval, $P_i(t_0)$ denotes the population in the region of origin at the beginning of the interval, and $P_j(t_1)$ represents the population in the region of destination at the end of the interval. Courgeau solves the resulting non-linear model for the P_i's, and he shows that in the case of three interacting regions, sustained cycles or even a chaotic behaviour may appear (depending on the values of the migration parameters) when the system is time-discrete. For the equivalent continuous-time system, the behaviour is much more regular[5]. An obvious extension of Courgeau's investigations would be to study to what extent the introduction of age increases or dampens this irregular behaviour.

Non-linear models may have great potential for the study of nuptiality (or, more generally, partnership formation), too. Instead of two or more interacting regions as in the previous example, we have here two interacting individuals. Two-dimensional marriage rates and probabilities, by ages of both spouses, involve the expression, in one denominator, of the time both spouses were exposed to the risk of marriage, or of the numbers of unmarried males and unmarried females. Traditional occurrence-exposure rates cannot be used. Proposals for "two-dimensional" rates and probabilities involve taking some average (arithmetic, geometric, harmonic mean) of the two exposure times or the two populations, or even more complicated functions, see, for example, Pollard

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5. Bartlett (1990, 324) notes that for continuous time systems, chaotic behaviour requires at least a three-dimensional phase space. Discrete-time systems can exhibit chaos in even one dimension, as is shown by the example of the logistic difference equation.

(1977). In general, such two–dimensional measures can be written as $H_{ij}/f(M_x,F_y)$, where H_{ij} is the number of marriages between males aged i and females aged j, and f is some scalar function of the vectors defined by unmarried males and females of all ages x and y, respectively (including, at least M_i and F_j). Most of these two–dimensional measures would imply a non–linear model for males and females broken down by marital status (married vs. not married), and possibly by age. It will be clear that the whole range of two–dimensional marriage rate specifications, in particular the function $f(M_x,F_y)$, defines a new programme of research for nuptiality and partnership behaviour, based upon the theory of non–linear models. This issue is closely linked to the two–sex problem, cf. below.

In studying non–linear demographic models, we should be careful, however, to balance possible tendencies to discern chaos everywhere when explanation and understanding fails, as Bartlett stressed. First, chaos and randomness may be considered as complementary. The study of stochastic processes has revealed many useful results, when for instance part of an unknown process is modelled as noise. Second, it will be clear that the inclusion of relevant factors in any demographic system is a very legitimate goal to be pursued. It cannot be denied, of course, that studies that attempt to explain the demographic behaviour of individuals or groups are still most relevant. However, at the same time we should realize that demographic behaviour cannot be fully explained and understood, however great the effort in demographic research will be. One of the consequences is that uncertainty in demographic predictions should be accepted, and can be accepted, cf. below.